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Non-differentiable Skew Convolution Semigroups and Related Ornstein-Uhlenbeck Processes

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Abstract: It is proved that a general non-differentiable skew convolution semigroup associated with a strongly continuous semigroup of linear operators on a real separable Hilbert space can be extended to a differentiable one on the entrance space of the linear semigroup. A càdlàg strong Markov process on an enlargement of the entrance space is constructed from which we obtain a realization of the corresponding Ornstein-Uhlenbeck process. Some explicit characterizations of the entrance spaces for special linear semigroups are given.

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1 Introduction

Suppose that $(S,+)$ is a Hausdorff topological semigroup and $(Q_t)_{t\geq 0}$ is a transition semigroup on S satisfying

$$Q_t(x_1+x_2,\cdot) = Q_t(x_1,\cdot) * Q_t(x_2,\cdot), \quad t\geq 0, x_1,x_2\in S, \tag{1.1}$$

where "$*$" denotes the convolution operation. A family of probability measures $(\mu_t)_{t\geq 0}$ on S is called a *skew convolution semigroup* (SC-semigroup) associated with $(Q_t)_{t\geq 0}$ if it satisfies

$$\mu_{r+t} = (\mu_r Q_t) * \mu_t, \quad r,t\geq 0. \tag{1.2}$$

This equation is of interest since it is satisfied if and only if

$$Q^\mu_t(x,\cdot) := Q_t(x,\cdot) * \mu_t(\cdot), \quad t\geq 0, x\in S, \tag{1.3}$$

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defines another transition semigroup $(Q_t^\mu)_{t\geq 0}$ on S. (Note that (1.1) implies $(\mu * \nu)Q_t = (\mu Q_t) * (\nu Q_t)$ for probability measures μ and ν on S.) This fact was first observed in [5, 6] when $S = M(E)$ is the space of all finite Borel measures on a metrizable space E; see also [8, Theorem 2.1]. In that case, $(Q_t)_{t\geq 0}$ corresponds to a measure-valued branching process and $(Q_t^\mu)_{t\geq 0}$ corresponds to an immigration process.

In this work, we shall consider the formulation in another special situation, where $S = H$ is a real separable Hilbert space and $Q_t(x, \cdot) \equiv \delta_{T_t x}$ for a strongly continuous semigroup of bounded linear operators $(T_t)_{t\geq 0}$ on H. In this case, we can rewrite (1.2) as

$$\mu_{r+t} = (T_t \mu_r) * \mu_t, \quad r, t \geq 0, \tag{1.4}$$

and the transition semigroup $(Q_t^\mu)_{t\geq 0}$ is given by

$$Q_t^\mu f(x) := \int_H f(T_t x + y)\mu_t(dy), \quad x \in H, f \in B(H), \tag{1.5}$$

where $B(H)$ denotes the totality of bounded Borel measurable functions on H. The semigroup $(Q_t^\mu)_{t\geq 0}$ defined by (1.5) is called a *generalized Mehler semigroup* associated with $(T_t)_{t\geq 0}$, which corresponds to a generalized Ornstein-Uhlenbeck process (OU-process). This definition of the generalized Mehler semigroup was given by Bogachev *et al* [1]. They also gave a characterization for the SC-semigroup $(\mu_t)_{t\geq 0}$ under the assumption that the function $t \mapsto \hat{\mu}_t(a)$ is differentiable at $t = 0$, where $\hat{\mu}_t(a)$ denotes the characteristic functional of μ_t. It is known that for a general SC-semigroup $(\mu_t)_{t\geq 0}$ defined by (1.4) the function $t \mapsto \hat{\mu}_t(a)$ is not necessarily differentiable at $t = 0$; see e.g. [**?**, 11, 12]. A simple and nice necessary and sufficient condition for an SC-semigroup to be differentiable was given in [11] in the setting of cylindrical probability measures. In [1] it was shown that a differentiable cylindrical Gaussian SC-semigroup can be extended into a real Gaussian SC-semigroup in an enlargement of H and the corresponding OU-process was constructed as the strong solution to a stochastic differential equation. Those results were extended to the general non-Gaussian case in [4]. A characterization for general SC-semigroups $(\mu_t)_{t\geq 0}$ was given in [**?**], where it was also observed that the OU-processes corresponding to a non-differentiable SC-semigroup usually have no right continuous realizations. This property is similar to that of the immigration processes studied in [6, 7, 8] and represents a departure from the theory of well-studied classes of OU-processes in [1, 4].

The main purpose of this paper is to study the construction of OU-processes corresponding to non-differentiable SC-semigroups. We shall see that, under a moment assumption, a general SC-semigroup can be decomposed as the convolution of a centered SC-semigroup and a degenerate one. For this reason, we shall only consider centered SC-semigroups. In section 2, we derive from the results of Dawson *et al* [**?**] that each centered SC-semigroup is uniquely determined by an infinitely divisible probability measure on the entrance space $\tilde{H}$ for the semigroup $(T_t)_{t\geq 0}$, which is an enlargement of H. In section 3, it is shown that a general non-differentiable centered SC-semigroup can always be extended to a differentiable one on the entrance space $\tilde{H}$. In section 4, we use a modification of the argument of Fuhrman and Röckner [4] to construct a càdlàg and strong Markov OU-process $\{\bar{X}_t : t \geq 0\}$ on a further extension $\bar{H}$ of $\tilde{H}$. We also show that, if $\bar{X}_0 \in H$, then $\bar{X}_t \in H$ almost surely for every $t \geq 0$ and $\{1_H(\bar{X}_t)\bar{X}_t : t \geq 0\}$ is an OU-process with transition semigroup $(Q_t^\mu)_{t\geq 0}$ defined by (1.5). Those results provide an approach to the study of non-differentiable generalized Mehler semigroups with which one can reduce some of their analysis to the existing framework of [1] and [4]. However, this approach

should not convince the reader that non-differentiable generalized Mehler semigroups do not bear particular consideration on their own. In fact, there are some cases where the natural state space of the OU-process is H and the introduction of $\tilde{H}$ and $\bar{H}$ seems unnatural and artificial. For example, an OU-process on $L^2(0,\infty)$ with non-differentiable SC-semigroup represents the fluctuation density of a catalytic branching processes with immigration; see [**?**]. In this case, it is rather unnatural to take $\bar{L}^2(0,\infty)$ as the state space. We provide some explicit characterization for the non-negative elements of $\tilde{L}^2(I\!R^d)$ and $\tilde{L}^2(0,\infty)$ in section 5. The explicit characterization for all elements of $\bar{L}^2(I\!R^d)$ and $\bar{L}^2(0,\infty)$ seems much more sophisticated.

2 Non-differentiable semigroups

Suppose that H is a real separable Hilbert space with dual space H^* and $(T_t)_{t\geq 0}$ is a strongly continuous semigroup of linear operators on H with dual $(T_t^*)_{t\geq 0}$. Let $(\mu_t)_{t\geq 0}$ be an SC-semigroup defined by (1.4) satisfying the moment condition

$$\int_{H^\circ} \|x\|^2 \mu_t(dx) < \infty, \quad t \geq 0, \tag{2.1}$$

where $H^\circ = H \setminus \{0\}$. Then we may define an H-valued path $(b_t)_{t\geq 0}$ by Bochner integrals

$$b_t := \int_{H^\circ} x\mu_t(dx), \quad t \geq 0,$$

and define $\mu_t^c = \delta_{-b_t} * \mu_t$. It is easy to check that both $(\delta_{b_t})_{t\geq 0}$ and $(\mu_t^c)_{t\geq 0}$ are SC-semigroups associated with $(T_t)_{t\geq 0}$ and $\mu_t = \mu_t^c * \delta_{b_t}$. That is, under the moment assumption, a general SC-semigroup can be decomposed as the convolution of a centered SC-semigroup and a degenerate one. For this reason, we shall only discuss centered SC-semigroups in the sequel.

Since $(T_t)_{t\geq 0}$ is strongly continuous, there are constants $c_0 \geq 0$ and $b_0 \geq 0$ such that $\|T_t\| \leq c_0 e^{b_0 t}$. Let $(U_\alpha)_{\alpha > b_0}$ denote the resolvent of $(T_t)_{t\geq 0}$ and let A denote its generator with domain $D(A) = U_\alpha H \subset H$. An H-valued path $\tilde{x} = \{\tilde{x}(s) : s > 0\}$ is called an *entrance path* for the semigroup $(T_t)_{t\geq 0}$ if it satisfies $\tilde{x}(r+t) = T_t\tilde{x}(r)$ for all $r, t > 0$. Let E denote the set of all entrance paths for $(T_t)_{t\geq 0}$. We say $\tilde{x} \in E$ is *closable* if there is an element $\tilde{x}(0) \in H$ such that $\tilde{x}(s) = T_s\tilde{x}(0)$ for all $s > 0$; and we say it is *locally square integrable* if

$$\int_0^l \|\tilde{x}(s)\|^2 ds < \infty \tag{2.2}$$

for some $l > 0$.

Lemma 2.1 *For any $\tilde{x} \in E$, (2.2) holds for some $l > 0$ if and only if it holds for all $l > 0$; and if and only if*

$$\int_0^\infty e^{-2bs} \|\tilde{x}(s)\|^2 ds < \infty \tag{2.3}$$

for all $b > b_0$.

Proof. Suppose that (2.2) holds for some $l_0 > 0$. Let $l > 0$ and let $n \geq 1$ be an integer such that $nl_0 \geq l$. Then

$$\begin{aligned}\int_0^l \|\tilde{x}(s)\|^2 ds &\leq \int_0^{nl_0} \|\tilde{x}(s)\|^2 ds \\ &= \sum_{k=0}^{n-1} \int_0^{l_0} \|T_{kl_0}\tilde{x}(s)\|^2 ds \\ &\leq \sum_{k=0}^{n-1} c_0^2 e^{2kb_0 l_0} \int_0^{l_0} \|\tilde{x}(s)\|^2 ds \\ &< \infty.\end{aligned}$$

Thus (2.2) holds for all $l > 0$. On the other hand, for any $b > b_0$,

$$\begin{aligned}\int_0^\infty e^{-2bs}\|\tilde{x}(s)\|^2 ds &= \sum_{k=0}^\infty e^{-2kbl_0} \int_0^{l_0} e^{-2bs}\|T_{kl_0}\tilde{x}(s)\|^2 ds \\ &\leq \sum_{k=0}^\infty c_0^2 e^{-2k(b-b_0)l_0} \int_0^{l_0} e^{-2bs}\|\tilde{x}(s)\|^2 ds \\ &< \infty.\end{aligned}$$

That is, (2.3) holds for all $b > b_0$. The remaining assertions are obvious. □

Let $\tilde{H}$ denote the set of all locally square integrable entrance paths for $(T_t)_{t\geq 0}$. We shall call $\tilde{H}$ the *entrance space* for $(T_t)_{t\geq 0}$. For any fixed $b > b_0$, we may define an inner product on $\tilde{H}$ by

$$\langle \tilde{x}, \tilde{y}\rangle_\sim := \int_0^\infty e^{-2bs}\langle \tilde{x}(s), \tilde{y}(s)\rangle ds, \quad \tilde{x}, \tilde{y} \in \tilde{H}. \tag{2.4}$$

Let $\|\cdot\|_\sim$ denote the norm induced by this inner product. The proof of the following result was suggested to us by W. Sun.

Lemma 2.2 *The normed space $(\tilde{H}, \|\cdot\|_\sim)$ is complete, so $(\tilde{H}, \langle\cdot,\cdot\rangle_\sim)$ is a Hilbert space.*

Proof. Suppose $\{\tilde{x}_n\} \subset \tilde{H}$ is a Cauchy sequence under the norm $\|\cdot\|_\sim$, that is,

$$\|\tilde{x}_n - \tilde{x}_m\|_\sim = \int_0^\infty e^{-2bs}\|\tilde{x}_n(s) - \tilde{x}_m(s)\|^2 ds \to 0$$

as $m, n \to \infty$. For each $t > 0$,

$$\begin{aligned}\|\tilde{x}_n(t) - \tilde{x}_m(t)\|^2 &= t^{-1}\int_0^t \|\tilde{x}_n(t) - \tilde{x}_m(t)\|^2 ds \\ &= t^{-1}\int_0^t \|T_{t-s}(\tilde{x}_n(s) - \tilde{x}_m(s))\|^2 ds \\ &\leq c_0^2 t^{-1} e^{2bt}\int_0^t e^{-2bs}\|\tilde{x}_n(s) - \tilde{x}_m(s)\|^2 ds.\end{aligned}$$

Then the limit $\tilde{x}(t) = \lim_{n\to\infty}\tilde{x}_n(t)$ exists in H. Since T_s is a continuous operator on H, for $s > 0$,

$$T_s\tilde{x}(t) = \lim_{n\to\infty} T_s\tilde{x}_n(t) = \lim_{n\to\infty}\tilde{x}_n(t+s) = \tilde{x}(t+s),$$

that is, $\tilde{x} = \{\tilde{x}(t) : t > 0\}$ is an entrance path for $(T_t)_{t\geq 0}$. For $\varepsilon > 0$, choose large enough $N \geq 1$ such that

$$\|\tilde{x}_n - \tilde{x}_m\|_\sim^2 = \int_0^\infty e^{-2bs}\|\tilde{x}_n(s) - \tilde{x}_m(s)\|^2 ds < \varepsilon$$

for $m, n \geq N$. By Fatou's lemma we get

$$\int_0^\infty e^{-2bs}\|\tilde{x}_n(s) - \tilde{x}(s)\|^2 ds \leq \liminf_{m\to\infty} \int_0^\infty e^{-2bs}\|\tilde{x}_n(s) - \tilde{x}_m(s)\|^2 ds \leq \varepsilon.$$

It follows that

$$\int_0^\infty e^{-2bs}\|\tilde{x}(s)\|^2 ds \leq \int_0^\infty e^{-2bs}\|\tilde{x}_n(s)\|^2 ds + \int_0^\infty e^{-2bs}\|\tilde{x}_n(s) - \tilde{x}(s)\|^2 ds < \infty.$$

Then $\tilde{x} \in \tilde{H}$ and $\lim_{n\to\infty} \|x_n - x\|_\sim^2 = 0$. $\square$

Lemma 2.3 *The map $J : x \mapsto \{T_s x : s > 0\}$ from $(H, \|\cdot\|)$ to $(\tilde{H}, \|\cdot\|_\sim)$ is a continuous dense embedding and hence $(\tilde{H}, \|\cdot\|_\sim)$ is separable.*

Proof. Since $x = \lim_{t\to 0^+} T_t x$, the map $J : x \mapsto \{T_s x : s > 0\}$ is injective. If $\lim_{n\to\infty} x_n = x \in H$, then

$$\int_0^\infty e^{-2bs}\|T_s x_n - T_s x\|^2 ds \leq c_0^2\|x_n - x\|^2 \cdot \int_0^\infty e^{-2(b-b_0)s} ds \to 0$$

as $n \to \infty$. Thus J is a continuous embedding. For an arbitrary $\tilde{x} \in \tilde{H}$ we have

$$\begin{aligned}
\|J\tilde{x}(t) - \tilde{x}\|_\sim^2 &= \int_0^\infty e^{-2bs}\|T_t\tilde{x}(s) - \tilde{x}(s)\|^2 ds \\
&= \int_0^r e^{-2bs}\|T_t\tilde{x}(s) - \tilde{x}(s)\|^2 ds + \int_r^\infty e^{-2bs}\|T_{s-r}[T_t\tilde{x}(r) - \tilde{x}(r)]\|^2 ds \\
&\leq 2(c_0^2 e^{2b_0 t} + 1)\int_0^r e^{-2bs}\|\tilde{x}(s)\|^2 ds \\
&\quad + c_0^2 e^{-2b_0 r}\|T_t\tilde{x}(r) - \tilde{x}(r)\|^2 \int_r^\infty e^{-2(b-b_0)s} ds.
\end{aligned}$$

Observe that the first integral on the right hand side goes to zero as $r \to 0^+$ and for fixed $r > 0$ the second term goes to zero as $t \to 0^+$. Then we have $\|J\tilde{x}(t) - \tilde{x}\|_\sim \to 0$ as $t \to 0$, and JH is dense in $\tilde{H}$. Since H is separable, so is $\tilde{H}$. $\square$

Theorem 2.1 *A family $(\mu_t)_{t\geq 0}$ of centered probability measures on H satisfying (2.1) is an SC-semigroup associated with $(T_t)_{t\geq 0}$ if and only if its characteristic functionals are given by*

$$\hat{\mu}_t(a) = \exp\left\{\int_0^t \log\hat{\nu}_s(a) ds\right\}, \quad t \geq 0, a \in H^*, \tag{2.5}$$

where $(\nu_s)_{s>0}$ is a family of centered infinitely divisible probability measures on H satisfying $\nu_{r+t} = T_t\nu_r$ for all $r, t > 0$ and

$$\int_0^t ds \int_{H^\circ} \|x\|^2 \nu_s(dx) < \infty, \quad t \geq 0, \tag{2.6}$$

and $\log\hat{\nu}_s(\cdot)$ denotes the unique continuous function on H^* with $\log\hat{\nu}_s(0) = 0$ and $\hat{\nu}_s(a) = \exp\{\log\hat{\nu}_s(a)\}$ for all $a \in H^*$.

Proof. It is well-known that the second moment of a centered infinitely divisible probability measure only involves the Gaussian covariance operator and the Lévy measure. If the centered probability measures $(\mu_t)_{t\geq 0}$ and $(\nu_s)_{s>0}$ are related by (2.5), the Gaussian covariance operators and Lévy measures of $(\mu_t)_{t\geq 0}$ can be represented as integrals of those of $(\nu_s)_{s>0}$. This observation yields that

$$\int_{H^\circ} \langle x, a\rangle^2 \mu_t(dx) = \int_0^t ds \int_{H^\circ} \langle x, a\rangle^2 \nu_s(dx), \quad t \geq 0, a \in H^*.$$

Let $\{e_n : n = 1, 2, \ldots\}$ be an orthonormal basis of $H = H^*$. Applying the above equation to each e_n and taking the summation we see

$$\int_{H^\circ} \|x\|^2 \mu_t(dx) = \int_0^t ds \int_{H^\circ} \|x\|^2 \nu_s(dx), \quad t \geq 0. \tag{2.7}$$

Thus conditions (2.1) and (2.6) are equivalent for the probability measures $(\mu_t)_{t\geq 0}$ and $(\nu_s)_{s>0}$ related by (2.5). Suppose $(\mu_t)_{t\geq 0}$ is given by (2.5) with the centered infinitely divisible probabilities $(\nu_s)_{s>0}$ satisfying $\nu_{r+t} = T_t\nu_r$ for all $r, t > 0$. Then $(\mu_t)_{t\geq 0}$ is a centered SC-semigroup by [**?**, Theorem 2.3]. Conversely, by [**?**, Theorems 2.1 and 2.2] any SC-semigroup $(\mu_t)_{t\geq 0}$ has the expression (2.5) up to the convolution of a family of degenerate probability measures $(\delta_{b_t})_{t\geq 0}$. If $(\mu_t)_{t\geq 0}$ is a centered SC-semigroup, we must have $b_t = 0$ for all $t \geq 0$. □

Theorem 2.2 *A family $(\mu_t)_{t\geq 0}$ of centered probability measures on H satisfying (2.1) is an SC-semigroup associated with $(T_t)_{t\geq 0}$ if and only if its characteristic functionals are given by*

$$\hat{\mu}_t(a) = \exp\Big\{ -\int_0^t \psi_s(a) ds\Big\}, \quad t \geq 0, a \in H^*, \tag{2.8}$$

where $\psi_s(\cdot)$ denotes the unique continuous function on H^* with $\psi_s(0) = 0$ and

$$\exp\{-\psi_s(a)\} = \int_{\tilde{H}} e^{i\langle \tilde{x}(s), a\rangle} \lambda_0(d\tilde{x}), \quad s > 0, a \in H^*, \tag{2.9}$$

where λ_0 is a centered infinitely divisible probability measure on $\tilde{H}$ satisfying

$$\int_{\tilde{H}} \|\tilde{x}\|_\sim^2 \lambda_0(d\tilde{x}) < \infty. \tag{2.10}$$

Proof. Let $(\nu_s)_{s>0}$ be given as in Theorem 2.1. In the terminology of Markov processes, $(\nu_s)_{s>0}$ is a probability entrance law for the Markov process $\{T_tx : t \geq 0\}$ with deterministic motion. Let $E_0 = H^{(0,\infty)}$ be the totality of paths $\{w(t) : t > 0\}$ from $(0, \infty)$ to H. We endow E_0 with the σ-algebra $\mathcal{E}_0$ generated by the maps $w \mapsto w(s)$, $s > 0$. By Kolmogorov's existence theorem, there is a unique probability measure λ_0 on E_0 so that $\{w(t) : t > 0\}$ under λ_0 is a Markov process with the same transition semigroup as the process $\{T_tx : t \geq 0\}$ and ν_s is the image of λ_0 under $w \mapsto w(s)$; see e.g. Sharpe [13, p.6]. Because of the special deterministic motion mechanism of $\{T_tx : t \geq 0\}$ we may assume that λ_0 is supported by the entrance paths E. Let $\mathcal{E}_0(E)$ and $\mathcal{E}_0(\tilde{H})$ denote respectively the traces of $\mathcal{E}_0$ on E and $\tilde{H}$. Since $w \mapsto \|w(s)\|^2$ is clearly a non-negative $\mathcal{E}_0(E)$-measurable function on E,

$$w \mapsto \|w\|_\sim := \int_0^\infty e^{-2bs} \|w(s)\|^2 ds$$

is an $\mathcal{E}_0(E)$-measurable function on E taking values in $[0,\infty]$. It is also easy to check that $\mathcal{E}_0(\tilde{H})$ coincides with the Borel σ-algebra $\mathcal{B}(\tilde{H})$ induced by the norm $\|\cdot\|_\sim$. Since $(\nu_s)_{s>0}$ satisfies (2.6), we have

$$\begin{aligned}
\int_E \|w\|_\sim^2 \lambda_0(dw) &= \int_E \lambda_0(dw)\int_0^\infty e^{-2bs}\|w(s)\|^2 ds \\
&= \int_0^\infty ds \int_H e^{-2bs}\|x\|^2 \nu_s(dx) \\
&= \sum_{n=0}^\infty \int_0^1 ds \int_H e^{-2b(n+s)}\|T_n x\|^2 \nu_s(dx) \\
&\leq c_0^2 \sum_{n=0}^\infty e^{-2(b-b_0)n}\int_0^1 ds \int_H e^{-2bs}\|x\|^2\nu_s(dx) \\
&< \infty,
\end{aligned}$$

so λ_0 is supported by $\tilde{H}$ and (2.10) holds. The infinite divisibility of λ_0 follows immediately from that of ν_s. $\square$

3 Differentiable extensions

For a general SC-semigroup given by Theorem 2.2, the function $t \mapsto \hat{\mu}_t(a)$ is not necessarily differentiable at $t=0$. However, if ν_0 is a centered infinitely divisible probability measure on H satisfying

$$\int_{H^\circ} \|x\|^2 \nu_0(dx) < \infty, \tag{3.1}$$

then

$$\hat{\mu}_t(a) = \exp\left\{\int_0^t \log \hat{\nu}_0(T_s^* a) ds\right\}, \quad t \geq 0, a \in H^*, \tag{3.2}$$

defines a centered SC-semigroup $(\mu_t)_{t\geq 0}$ such that $t \mapsto \hat{\mu}_t(a)$ is differentiable at $t=0$ for all $a \in H^*$. In the sequel, we shall call $(\mu_t)_{t\geq 0}$ a *differentiable* SC-semigroup if it is given by (3.2). We shall discuss how to extend a general SC-semigroup on H to a differentiable one on the entrance space $\tilde{H}$. For any strongly continuous linear semigroup $(T_t)_{t\geq 0}$ on H,

$$(\tilde{T}_t\tilde{x})(s) = \tilde{x}(t+s), \quad s,t>0, \tag{3.3}$$

defines a semigroup of linear operators $(\tilde{T}_t)_{t\geq 0}$ on $\tilde{H}$. Since

$$\|\tilde{T}_t\tilde{x}\|_\sim^2 = \int_0^\infty e^{-2bs}\|\tilde{x}(t+s)\|^2 ds \leq \|T_t\|^2 \int_0^\infty e^{-2bs}\|\tilde{x}(s)\|^2 ds,$$

we have $\|\tilde{T}_t\|_\sim \leq \|T_t\|$. Let $(\tilde{U}_\alpha)_{\alpha>b_0}$ denote the resolvent of $(\tilde{T}_t)_{t\geq 0}$ and let $\tilde{A}$ denote its generator with domain $D(\tilde{A}) = \tilde{U}_\alpha\tilde{H} \subset \tilde{H}$.

Lemma 3.1 *Let J be defined as in Lemma 2.3. Then $JT_t x = \tilde{T}_t Jx$ for all $t \geq 0$ and $x \in H$ and $(\tilde{T}_t)_{t\geq 0}$ is a strongly continuous semigroup of linear operators on $\tilde{H}$.*

Proof. For $t \geq 0$ and $x \in H$ we have

$$JT_t x = \{T_s T_t x : s > 0\} = \{T_t T_s x : s > 0\} = \tilde{T}_t Jx,$$

giving the first assertion. By the proof of Lemma 2.3, $\|\tilde{T}_t \tilde{x} - \tilde{x}\|_\sim = \|J\tilde{x}(t) - \tilde{x}\|_\sim \to 0$ as $t \to 0$, that is, $(\tilde{T}_t)_{t\geq 0}$ is strongly continuous. $\square$

Lemma 3.2 *We have $\tilde{U}_\alpha \tilde{x} = \{U_\alpha \tilde{x}(s) : s > 0\}$ and $\tilde{A}\tilde{U}_\alpha \tilde{x} = \{AU_\alpha x(s) : s > 0\}$ for all $\tilde{x} \in \tilde{H}$.*

Proof. The first assertion follows as we observe that

$$\tilde{U}_\alpha \tilde{x}(s) = \int_0^\infty e^{-\alpha s} \tilde{T}_t \tilde{x}(s) dt = \int_0^\infty e^{-\alpha s} T_t \tilde{x}(s) dt = U_\alpha \tilde{x}(s),$$

and the second follows from the equality $\tilde{A}\tilde{U}_\alpha \tilde{x} = \alpha \tilde{U}_\alpha \tilde{x} - \tilde{x}$. $\square$

Theorem 3.1 *All centered SC-semigroups associated with $(T_t)_{t\geq 0}$ satisfying (2.1) are differentiable if and only if all its locally square integrable entrance paths are closable.*

Proof. Suppose that all entrance paths $\tilde{x} \in \tilde{H}$ are closable and $(\mu_t)_{t\geq 0}$ is an SC-semigroup given by (2.8). To each $\tilde{x} \in \tilde{H}$ there corresponds some $\tilde{x}(0) \in H$ such that $\tilde{x}(s) = T_s \tilde{x}(0)$ for all $s > 0$. This element is apparently determined by $\tilde{x}$ uniquely. Letting ν_0 be the image of λ_0 under the map $\tilde{x} \mapsto \tilde{x}(0)$ we get (3.2). Conversely, if $\tilde{x} = \{\tilde{x}(s) : s > 0\} \in \tilde{H}$ is not closable, then

$$\hat{\mu}_t(a) = \exp\left\{ -\frac{1}{2} \int_0^t \langle \tilde{x}(s), a\rangle^2 ds \right\}, \quad t \geq 0, a \in H^*,$$

defines a non-differentiable SC-semigroup. $\square$

Theorem 3.2 *All entrance paths for $(\tilde{T}_t)_{t\geq 0}$ are closable.*

Proof. Suppose that $\bar{x} = \{\bar{x}(u) : u > 0\}$ is an entrance path for $(\tilde{T}_t)_{t\geq 0}$, where each $\bar{x}(u) = \{\bar{x}(u,s) : s > 0\} \in \tilde{H}$ is an entrance path for $(T_t)_{t\geq 0}$. Then we get

$$\{\bar{x}(u, r+s) : s > 0\} = (\tilde{T}_r \bar{x})(u) = \bar{x}(r+u) = \{\bar{x}(r+u, s) : s > 0\}, \tag{3.4}$$

where the first equality follows from (3.3) and the second one holds since $\bar{x}$ is an entrance path for $(\tilde{T}_t)_{t\geq 0}$. Setting $\bar{x}(0) = \{\bar{x}(s/2, s/2) : s > 0\}$ we have

$$\tilde{T}_u \bar{x}(0)(s) = \bar{x}(s/2, u + s/2) = \bar{x}(u, s), \tag{3.5}$$

where the first equality follows from (3.3) and the second one holds by (3.4). Thus $\tilde{T}_u \bar{x}(0) = \bar{x}(u)$, that is, $\bar{x} = \{\bar{x}(u) : u > 0\}$ is closed by $\bar{x}(0)$. $\square$

For the infinitely divisible probability measure λ_0 on $\tilde{H}$ given by Theorem 2.2 we have

$$\hat{\lambda}_0(\tilde{a}) = e^{-\tilde{\psi}_0(\tilde{a})}, \quad \tilde{a} \in \tilde{H}^*, \tag{3.6}$$

for a functional $\tilde{\psi}_0$ on $\tilde{H}^*$ with representation

$$\tilde{\psi}_0(\tilde{a}) = \frac{1}{2}\langle \tilde{R}a, a\rangle_\sim - \int_{\tilde{H}^\circ} \left(e^{i\langle \tilde{x}, \tilde{a}\rangle_\sim} - 1 - i\langle \tilde{x}, \tilde{a}\rangle_\sim \right) \tilde{M}(d\tilde{x}), \quad \tilde{a} \in \tilde{H}^*, \tag{3.7}$$

where R is a nuclear operator on $\tilde{H}$ and $\|\tilde{x}\|_\sim^2 \tilde{M}(d\tilde{x})$ is a finite measure on $\tilde{H}^\circ := \tilde{H} \setminus \{0\}$; see e.g. [10].

Theorem 3.3 *Let $(\mu_t)_{t\geq 0}$ be a centered SC-semigroup given by (2.8) and let $\tilde{\mu}_t = J\mu_t$. Then $(\tilde{\mu}_t)_{t\geq 0}$ is a differentiable centered SC-semigroup associated with $(\tilde{T}_t)_{t\geq 0}$ and*

$$\int_{\tilde{H}} e^{i\langle \tilde{x},\tilde{a}\rangle_\sim}\tilde{\mu}_t(d\tilde{x}) = \exp\Big\{-\int_0^t \tilde{\psi}_0(\tilde{T}_s^*\tilde{a})ds\Big\}, \quad t\geq 0, \tilde{a}\in\tilde{H}^*. \tag{3.8}$$

Proof. By Lemma 2.3, $J : H \mapsto \tilde{H}$ is an embedding. Thus $(\tilde{\mu}_t)_{t\geq 0}$ is an SC-semigroup associated with $(\tilde{T}_t)_{t\geq 0}$. Since $(T_t)_{t\geq 0}$ is a strongly continuous semigroup, for any $\tilde{a} = \{\tilde{a}(s) : s > 0\} \in \tilde{H}$ we have by dominated convergence

$$\begin{aligned}
&\int_H \exp\Big\{i\int_0^\infty e^{-2bs}\langle T_s x, \tilde{a}(s)\rangle ds\Big\}\mu_t(dx)\\
=\;& \lim_{n\to\infty}\int_H \exp\Big\{i\sum_{k=1}^\infty n^{-1}e^{-2bk/n}\langle T_{k/n}x, \tilde{a}(k/n)\rangle\Big\}\mu_t(dx)\\
=\;& \lim_{n\to\infty}\int_H \exp\Big\{i\Big\langle x, \sum_{k=1}^\infty n^{-1}e^{-2bk/n}T_{k/n}^*\tilde{a}(k/n)\Big\rangle\Big\}\mu_t(dx)\\
=\;& \lim_{n\to\infty}\exp\Big\{\int_0^t\Big[\log\int_H \exp\Big\{i\Big\langle x, \sum_{k=1}^\infty n^{-1}e^{-2bk/n}T_{k/n}^*\tilde{a}(k/n)\Big\rangle\Big\}\nu_s(dx)\Big]ds\Big\}\\
=\;& \lim_{n\to\infty}\exp\Big\{\int_0^t\Big[\log\int_H \exp\Big\{i\sum_{k=1}^\infty n^{-1}e^{-2bk/n}\langle T_{k/n}x, \tilde{a}(k/n)\rangle\Big\}\nu_s(dx)\Big]ds\Big\}\\
=\;& \exp\Big\{\int_0^t\Big[\log\int_H \exp\Big\{i\int_0^\infty e^{-2bu}\langle T_u x, \tilde{a}(u)\rangle du\Big\}\nu_s(dx)\Big]ds\Big\}.
\end{aligned}$$

It follows that

$$\int_{\tilde{H}} e^{i\langle \tilde{x},\tilde{a}\rangle_\sim}J\mu_t(d\tilde{x}) = \exp\Big\{\int_0^t\Big[\log\int_{\tilde{H}} e^{i\langle \tilde{x},\tilde{a}\rangle_\sim}J\nu_s(d\tilde{x})\Big]ds\Big\}, \quad t\geq 0, \tilde{a}\in\tilde{H}. \tag{3.9}$$

Recall from the proof of Theorem 2.2 that ν_s is the image of λ_0 under $\tilde{x}\mapsto\tilde{x}(s)$. Then $\tilde{T}_s\lambda_0 = J\nu_s$ and (3.8) follows from (3.9) and (3.6). □

Theorem 3.4 *Let $(\tilde{\mu}_t)_{t\geq 0}$ be a centered SC-semigroup associated with $(\tilde{T}_t)_{t\geq 0}$ satisfying*

$$\int_{\tilde{H}^\circ}\|\tilde{x}\|^2\tilde{\mu}_t(d\tilde{x}) < \infty, \quad t\geq 0. \tag{3.10}$$

Then there is a centered SC-semigroup $(\mu_t)_{t\geq 0}$ associated with $(T_t)_{t\geq 0}$ satisfying (2.1) and $\tilde{\mu}_t = J\mu_t$ for each $t\geq 0$.

Proof. By Theorems 3.1 and 3.2, $(\tilde{\mu}_t)_{t\geq 0}$ is differentiable, so it has the expression (3.8) for an infinitely divisible probability λ_0 on $\tilde{H}$ defined by (3.6). Then we get $(\mu_t)_{t\geq 0}$ by Theorem 2.2, which clearly satisfies the requirements. □

By Theorems 3.3 and 3.4, centered SC-semigroups associated with $(T_t)_{t\geq 0}$ and those associated with $(\tilde{T}_t)_{t\geq 0}$ are in 1-1 correspondence. Therefore we may reduce some analysis of non-differentiable centered SC-semigroups to those of differentiable ones studied in [1, 4].

4 Ornstein-Uhlenbeck processes

In this section, we discuss constructions of the OU-processes. By the results of the last section, a general centered SC-semigroup on H can be extended to a differentiable one on the entrance space $\tilde{H}$. Then by Fuhrman and Röckner [4, Theorem 5.3], there is an extension E of $\tilde{H}$ on which a càdlàg realization of the corresponding OU-process can be constructed. In the sequel, we shall give a modification of the arguments of Fuhrman and Röckner [4] which provides a smaller extension but still contains a càdlàg realization of the OU-process. Fix $\alpha > b_0$ and define an inner product on $\tilde{H}$ by

$$\langle \tilde{x}, \tilde{y} \rangle_- = \int_0^\infty e^{-2bs} \langle U_\alpha \tilde{x}(s), U_\alpha \tilde{y}(s) \rangle ds, \quad \tilde{x}, \tilde{y} \in \tilde{H}. \tag{4.1}$$

Let $\|\cdot\|_-$ be the corresponding norm and let $\bar{H}$ be the completion of $\tilde{H}$ with respect to $\|\cdot\|_-$.

Lemma 4.1 *For $\tilde{x} \in \tilde{H}$ we have $\|\tilde{x}\|_- \le \|U_\alpha\| \|\tilde{x}\|_\sim$, so the identity mapping I from $(\tilde{H}, \|\cdot\|_\sim)$ to $(\bar{H}, \|\cdot\|_-)$ is a continuous embedding.*

Proof. By (2.4) and (4.1),

$$\|\tilde{x}\|_-^2 = \int_0^\infty e^{-2bs} \|U_\alpha \tilde{x}(s)\|^2 ds \le \|U_\alpha\|^2 \int_0^\infty e^{-2bs} \|\tilde{x}(s)\|^2 ds = \|U_\alpha\|^2 \|\tilde{x}\|_\sim^2,$$

giving the desired estimate. □

Note that the embedding of $(\tilde{H}, \|\cdot\|_\sim)$ into $(\bar{H}, \|\cdot\|_-)$ is not necessarily Hilbert-Schmidt, so our extension is different from the one used in [4]. For $\tilde{x} \in \tilde{H}$ we have, by (4.1),

$$\|\tilde{T}_t \tilde{x}\|_-^2 = \int_0^\infty e^{-2bs} \|U_\alpha T_t \tilde{x}(s)\|^2 ds = \int_0^\infty e^{-2bs} \|U_\alpha \tilde{x}(t+s)\|^2 ds \le e^{2bt} \|\tilde{x}\|_-^2.$$

Then each $\tilde{T}_t$ has a unique extension to a bounded linear operator $\bar{T}_t$ on $\bar{H}$. Since the semigroup property and strong continuity of $(\bar{T}_t)_{t\ge0}$ hold on the dense subspace $\tilde{H}$ of $\bar{H}$, they also hold on $\bar{H}$, that is, the semigroup of linear operators $(\bar{T}_t)_{t\ge0}$ extends $(\tilde{T}_t)_{t\ge0}$. Let $(\bar{U}_\alpha)_{\alpha>b_0}$ denote the resolvent of $(\bar{T}_t)_{t\ge0}$ and let $\bar{A}$ denote its generator with domain $D(\bar{A}) = \bar{U}_\alpha \bar{H} \subset \bar{H}$. Then $D(\bar{A})$ is a Hilbert space with inner product norm $\|\cdot\|_{\bar{A}}$ defined by

$$\|\bar{x}\|_{\bar{A}} = \|\bar{x}\|_- + \|\bar{A}\bar{x}\|_-, \quad \bar{x} \in D(\bar{A}). \tag{4.2}$$

Lemma 4.2 *We have $\tilde{H} \subset D(\bar{A})$ and*

$$\|\bar{A}\tilde{y}\|_- \le 2(\alpha^2 \|U_\alpha\|^2 + 1)^{1/2} \|\tilde{y}\|_\sim, \quad \tilde{y} \in \tilde{H}. \tag{4.3}$$

Consequently, the identity mapping I from $(\tilde{H}, \|\cdot\|_\sim)$ to $(D(\bar{A}), \|\cdot\|_{\bar{A}})$ is a continuous embedding.

Proof. Suppose that $\tilde{y} \in D(\tilde{A}) \subset D(\bar{A})$. Then $\tilde{y} = \tilde{U}_\alpha \tilde{x}$ for some $\tilde{x} \in \tilde{H}$. By (4.1) and Lemma 3.2,

$$\begin{aligned}
\|\bar{A}\tilde{y}\|_-^2 &= \|\tilde{A}\tilde{U}_\alpha \tilde{x}\|_-^2 \\
&= \int_0^\infty e^{-2bs} \|AU_\alpha^2 \tilde{x}(s)\|^2 ds \\
&= \int_0^\infty e^{-2bs} \|\alpha U_\alpha^2 \tilde{x}(s) - U_\alpha \tilde{x}(s)\|^2 ds \\
&\le 2(\alpha^2 \|U_\alpha\|^2 + 1) \int_0^\infty e^{-2bs} \|U_\alpha \tilde{x}(s)\|^2 ds \\
&= 2(\alpha^2 \|U_\alpha\|^2 + 1) \|\tilde{y}\|_\sim^2.
\end{aligned}$$

Since $D(\tilde{A})$ is a dense subset of $(\tilde{H}, \|\cdot\|_\sim)$, we have $\tilde{H} \subset D(\bar{A})$ by (4.2) and the above inequality remains true for all $\tilde{y} \in \tilde{H}$. $\square$

Now suppose that $(\mu_t)_{t\geq 0}$ and $(\tilde{\mu}_t)_{t\geq 0}$ are the SC-semigroups described in Theorem 3.3. Let $\bar{\mu}_t$ be the unique probability measure on $\bar{H}$ whose restriction to $\tilde{H}$ is $\tilde{\mu}_t$. Then $(\bar{\mu}_t)_{t\geq 0}$ is an SC-semigroup associated with $(\bar{T}_t)_{t\geq 0}$. By (3.8),

$$\int_{\bar{H}} e^{i\langle \bar{x}, \bar{a}\rangle_-} \bar{\mu}_t(d\bar{x}) = \exp\bigg\{ -\int_0^t \tilde{\psi}_0(\bar{T}_s^* \bar{a}) ds\bigg\}, \quad t \geq 0, \bar{a} \in \bar{H}^* \subset \tilde{H}^*, \tag{4.4}$$

where $\tilde{\psi}_0(\cdot)$ is defined by (3.6). Let $(Q_t^{\bar{\mu}})_{t\geq 0}$ be the generalized Mehler semigroup defined by (1.5) from $(\bar{T}_t)_{t\geq 0}$ and $(\bar{\mu}_t)_{t\geq 0}$. By [4, Theorem 5.1], there is a càdlàg $\tilde{H}$-valued process $\{\tilde{Y}_t : t \geq 0\}$ with $\tilde{Y}_0 = 0$ and with independent increments such that $\tilde{Y}_t - \tilde{Y}_r$ has distribution γ_{t-r} with

$$\hat{\gamma}_{t-r}(\tilde{a}) = \exp\{-(t-r)\tilde{\psi}_0(\tilde{a})\}, \quad t \geq r \geq 0, \tilde{a} \in \tilde{H}^*. \tag{4.5}$$

By the strong continuity of $(\bar{T}_t)_{t\geq 0}$ and Lemma 4.2, $s \mapsto \bar{T}_{t-s}\bar{A}\tilde{Y}_s$ is a right continuous $\bar{H}$-valued function of $s \in [0, t]$. Then for any given $\bar{x} \in \bar{H}$ we may define the càdlàg $\bar{H}$-valued process $\{\bar{X}_t : t \geq 0\}$ by

$$\bar{X}_t = \bar{T}_t\bar{x} + \tilde{Y}_t + \int_0^t \bar{T}_{t-s}\bar{A}\tilde{Y}_s ds, \quad t \geq 0. \tag{4.6}$$

Lemma 4.3 *The $\bar{H}$-valued random variable $\bar{X}_t$ has distribution $Q_t^{\bar{\mu}}(\bar{x}, \cdot)$ for every $t \geq 0$. In particular, if $\bar{x} \in JH$, then $\bar{X}_t \in JH$ a.s. for every $t \geq 0$.*

Proof. We first prove that $\bar{X}_t^{(0)} := \bar{X}_t - \bar{T}_t\bar{x}$ has distribution $\bar{\mu}_t(\cdot) = Q_t^{\bar{\mu}}(0, \cdot)$. By the right continuity of $s \mapsto \bar{T}_{t-s}\bar{A}\tilde{Y}_s$, we have

$$\bar{X}_t^{(n)} := \tilde{Y}_t + \frac{t}{n}\sum_{k=1}^n \bar{T}_{(1-k/n)t}\bar{A}\tilde{Y}_{kt/n} \to \tilde{Y}_t + \int_0^t \bar{T}_{t-s}\bar{A}\tilde{Y}_s ds = \bar{X}_t^{(0)}$$

as $n \to \infty$. Let $D_0 = 0$ and $D_k = \bar{T}_{(1-n/n)t}\bar{A} + \cdots + \bar{T}_{(1-k/n)t}\bar{A}$. Then we have

$$\begin{aligned}
\bar{X}_t^{(n)} &= \tilde{Y}_t + n^{-1}t[(D_1 - D_2)\tilde{Y}_{t/n} + \cdots + (D_{n-1} - D_n)\tilde{Y}_{(n-1)t/n} + D_n\tilde{Y}_{nt/n}] \\
&= (\tilde{Y}_{nt/n} - \tilde{Y}_{(n-1)t/n}) + \cdots + (\tilde{Y}_{2t/n} - \tilde{Y}_{t/n}) + (\tilde{Y}_{t/n} - \tilde{Y}_0) \\
&\quad + n^{-1}t[D_1(\tilde{Y}_{t/n} - \tilde{Y}_0) + D_2(\tilde{Y}_{2t/n} - \tilde{Y}_{t/n}) + \cdots \\
&\quad + D_n(\tilde{Y}_{nt/n} - \tilde{Y}_{(n-1)t/n})] \\
&= (I + n^{-1}tD_1)(\tilde{Y}_{t/n} - \tilde{Y}_0) + (I + n^{-1}tD_2)(\tilde{Y}_{2t/n} - \tilde{Y}_{t/n}) + \cdots \\
&\quad + (I + n^{-1}tD_n)(\tilde{Y}_{nt/n} - \tilde{Y}_{(n-1)t/n})].
\end{aligned}$$

It follows that

$$\begin{aligned}
\boldsymbol{E}\exp\Big\{i\langle \bar{X}_t^{(n)}, \bar{a}\rangle_-\Big\} &= \boldsymbol{E}\exp\bigg\{i\sum_{k=1}^n \langle (I + n^{-1}tD_k)(\tilde{Y}_{kt/n} - \tilde{Y}_{(k-1)t/n}), \bar{a}\rangle_-\bigg\} \\
&= \boldsymbol{E}\exp\bigg\{i\sum_{k=1}^n \langle (\tilde{Y}_{kt/n} - \tilde{Y}_{(k-1)t/n}), (I + n^{-1}tD_k)^*\bar{a}\rangle_-\bigg\} \\
&= \exp\bigg\{ -\frac{t}{n}\sum_{k=1}^n \tilde{\psi}_0((I + n^{-1}tD_k)^*\bar{a})\bigg\}.
\end{aligned}$$

In view of (3.7), $\tilde{\psi}_0(\cdot)$ is uniformly continuous on any bounded subset of $\bar{H}^*$. Observe also that

$$\begin{aligned}
&\|\bar{T}^*_{(1-k/n)t}\bar{a} - (I + n^{-1}tD_k)^*\bar{a}\|_- \\
= &\left\|\bar{T}^*_{(1-k/n)t}\bar{a} - \bar{a} - \frac{t}{n}\sum_{j=k}^{n}\bar{T}^*_{(1-j/n)t}\bar{A}^*\bar{a}\right\|_{\bar{H}} \\
\leq &\sum_{j=k}^{n}\int_{(1-j/n)t}^{(1-(j-1)/n)t}\|\bar{T}^*_s\bar{A}^*\bar{a} - \bar{T}^*_{(1-j/n)t}\bar{A}^*\bar{a}\|_- ds \\
\leq &\ t\cdot\sup\left\{\|\bar{T}^*_{t_2}\bar{A}^*\bar{a} - \bar{T}^*_{t_1}\bar{A}^*\bar{a}\|_- : 0 \leq t_1, t_2 \leq t \text{ and } |t_2 - t_1| < t/n\right\},
\end{aligned}$$

which goes to zero as $n \to \infty$. Thus we have

$$\begin{aligned}
\boldsymbol{E}\exp\left\{i\langle\bar{X}^{(0)}_t, \bar{a}\rangle_-\right\} &= \lim_{n\to\infty}\boldsymbol{E}\exp\left\{i\langle\bar{X}^{(n)}_t, \bar{a}\rangle_-\right\} \\
&= \lim_{n\to\infty}\exp\left\{-\frac{t}{n}\sum_{k=1}^{n}\tilde{\psi}_0(\bar{T}^*_{(1-k/n)t}\bar{a})\right\} \\
&= \exp\left\{-\int_0^t\tilde{\psi}_0(\bar{T}^*_{t-s}\bar{a})ds\right\},
\end{aligned}$$

so that $\bar{X}^{(0)}_t$ has distribution $Q^{\bar{\mu}}_t(0,\cdot)$. Therefore, $\bar{X}_t$ has distribution $Q^{\bar{\mu}}_t(\bar{x},\cdot)$. If $\bar{x} = Jx$ for some $x \in H$, then $\bar{T}_t\bar{x} = \tilde{T}_tJx = JT_tx \in JH$ by Lemma 3.1. Since $\bar{\mu}_t(\cdot)$ is supported by JH, so is $Q^{\bar{\mu}}_t(\bar{x},\cdot)$ and hence a.s. $\bar{X}_t \in JH$. □

Theorem 4.1 *The process $\{\bar{X}_t : t \geq 0\}$ defined by (4.6) is a càdlàg strong Markov process with transition semigroup $(Q^{\bar{\mu}}_t)_{t\geq 0}$.*

Proof. By the construction (4.6), $\{\bar{X}_t : t \geq 0\}$ is adapted to the filtration $\mathscr{F}_t := \sigma(\{\tilde{Y}_s : 0 \leq s \leq t\})$. For $r, t \geq 0$,

$$\begin{aligned}
\bar{X}_{r+t} - \bar{T}_t\bar{X}_r &= \tilde{Y}_{r+t} - \bar{T}_t\tilde{Y}_r + \int_r^{r+t}\bar{T}_{r+t-s}\bar{A}\tilde{Y}_s ds \\
&= (\tilde{Y}_{r+t} - \tilde{Y}_r) + \int_r^{r+t}\bar{T}_{r+t-s}\bar{A}(\tilde{Y}_s - \tilde{Y}_r)ds.
\end{aligned}$$

Since $\{\tilde{Y}_{r+t} - \tilde{Y}_r : t \geq 0\}$ given $\mathscr{F}_r$ is a process with independent increments and has the same law as $\{\tilde{Y}_t : t \geq 0\}$, Lemma 4.3 implies that

$$\boldsymbol{E}\left[\exp\{i\langle\bar{X}_{r+t}, \bar{a}\rangle_-\}\,\middle|\,\mathscr{F}_r\right] = \exp\left\{i\langle\bar{X}_r, \bar{T}^*_t\bar{a}\rangle_- - \int_0^t\tilde{\psi}_0(\bar{T}^*_s\bar{a})ds\right\}.$$

Thus $\{\bar{X}_t : t \geq 0\}$ is a Markov process with transition semigroup $(Q^{\bar{\mu}}_t)_{t\geq 0}$. The strong Markov property holds since $(Q^{\bar{\mu}}_t)_{t\geq 0}$ is Feller. □

Now let $\bar{x} = Jx$ for some $x \in H$. In this case, $\bar{X}_t \in JH$ a.s. by Lemma 4.3. Recall that $J : H \to JH \subset \tilde{H} \subset \bar{H}$ and let $X_t = 1_{JH}(\bar{X}_t)J^{-1}(\bar{X}_t)$, where $J^{-1} : JH \to H$ denotes the inverse map of J. Then $\{X_t : t \geq 0\}$ is an OU-process with transition semigroup $(Q^{\mu}_t)_{t\geq 0}$ and $X_0 = x$. This gives a construction of the OU-process $\{X_t : t \geq 0\}$ from the càdlàg strong Markov process $\{\bar{X}_t : t \geq 0\}$. In general, $\{X_t : t \geq 0\}$ does not have right continuous modification in H. A similar construction in the measure-valued setting has been used in [6] to prove the non-existence of right continuous realization of a general immigration process.

5 Brownian transition semigroups

We have seen that a general SC-semigroup on H can always be extended to a differentiable one in the entrance space $\tilde{H}$ and a càdlàg realization of the corresponding OU-process can always be constructed in an extension $\bar{H}$ of $\tilde{H}$. In this section, we provided some explicit characterization for the non-negative elements of $\tilde{L}^2(I\!R^d)$ and $\tilde{L}^2(0,\infty)$ constructed respectively from $L^2(I\!R^d)$ and $L^2(0,\infty)$. It seems that the explicit characterization for all elements of $\bar{L}^2(I\!R^d)$ and $\bar{L}^2(0,\infty)$ is much more sophisticated.

We first consider the case where $(T_t)_{t\geq 0}$ is the transition semigroup of the standard Brownian motion on $I\!R^d$. Let $\tilde{H} := \tilde{L}^2(I\!R^d)$ be defined from $H := L^2(I\!R^d)$ and $(T_t)_{t\geq 0}$. Let

$$g_d(s,x) = \frac{1}{(2\pi s)^{d/2}} \exp\{-|x|^2/2s\}, \quad s>0, x\in I\!R^d, \tag{5.1}$$

where $|\cdot|$ denote the Euclidean norm on $I\!R^d$, and let $S(I\!R^d)$ be the set of signed-measures μ on $I\!R^d$ with total variation measures $\|\mu\|$ satisfying

$$\int_0^l ds \int_{I\!R^d} \|\mu\|(dx) \int_{I\!R^d} g_d(2s, y-x)\|\mu\|(dy) < \infty \tag{5.2}$$

for some $l>0$. Let $S_+(I\!R^d)$ and $\tilde{L}^2_+(I\!R^d)$ denote respectively the subsets of non-negative elements of $S(I\!R^d)$ and $\tilde{L}^2(I\!R^d)$.

Theorem 5.1 *There is a 1-1 correspondence between $\tilde{x}\in\tilde{L}^2_+(I\!R^d)$ and $\mu\in S_+(I\!R^d)$ which is given by*

$$\tilde{x}(s,\cdot) = \int_{I\!R^d} g_d(s,\cdot - z)\mu(dz), \quad s>0. \tag{5.3}$$

Proof. If $\mu\in S_+(I\!R^d)$, then (5.3) defines a non-negative entrance path $\tilde{x}$ for $(T_t)_{t\geq 0}$. Since

$$\begin{aligned}
\int_0^l \|\tilde{x}(s,\cdot)\|^2 ds &= \int_0^l ds \int_{I\!R^d} \Big(\int_{I\!R^d} g_d(s,y-z)\mu(dz)\Big)^2 dy \\
&= \int_0^l ds \int_{I\!R^d} dy \int_{I\!R^d} g_d(s,y-x)\mu(dx) \int_{I\!R^d} g_s(y-z)\mu(dz) \\
&= \int_0^l ds \int_{I\!R^d} \mu(dx) \int_{I\!R^d} g_d(2s,z-x)\mu(dz) \\
&< \infty,
\end{aligned}$$

we have $\tilde{x}\in\tilde{L}^2_+(I\!R^d)$. Conversely, suppose that $\tilde{x}\in\tilde{L}^2_+(I\!R^d)$ and let $\kappa_s(dy) = \tilde{x}(s,y)dy$. Then $(\kappa_s)_{s>0}$ is a measure-valued entrance law for $(T_t)_{t\geq 0}$. By the property of the Brownian semigroup, there is a measure μ on $I\!R^d$ such that $\kappa_s = \mu T_s$; see e.g. Dynkin [3, p.80]. Thus $\tilde{x}(s,\cdot)$ has the representation (5.3), and (5.2) follows from (2.2) and the calculations above. $\square$

When $d=1$, we can give a necessary and sufficient condition for (5.2). Observe that for $0<l\leq 1$ we have

$$\int_0^l g_1(2s,y-x)ds < \int_0^1 \frac{1}{2\sqrt{\pi s}} \exp\{-(y-x)^2/4\}ds = \frac{1}{\sqrt{\pi}} \exp\{-(y-x)^2/4\},$$

and for $l > 1$ we have

$$\int_0^l g_1(2s, y-x)ds > \int_1^l \frac{1}{2\sqrt{\pi l}} \exp\{-(y-x)^2/4\}ds = \frac{l-1}{2\sqrt{\pi l}} \exp\{-(y-x)^2/4\}.$$

By Lemma 2.1 and the proof of Theorem 5.1, (5.2) holds if and only if

$$\int_{I\!R} \|\mu\|(dx) \int_{I\!R} \exp\{-(y-x)^2/4\} \|\mu\|(dy) < \infty. \tag{5.4}$$

Theorem 5.1 gives a complete characterization of non-negative elements of $\tilde{L}^2(I\!R^d)$. By this result, (5.3) also defines an element of $\tilde{L}^2(I\!R^d)$ for $\mu \in S(I\!R^d)$. Unfortunately, this representation does not give all elements of $\tilde{L}^2(I\!R^d)$. To see this, take any sequence $\{a_k\} \subset I\!R$ and observe that

$$\int_0^\infty e^{-2bs} ds \int_{I\!R} [g_1(s, y-x) - g_1(s,y)]^2 dy \to 0$$

as $x \to 0$. Then for each $k \geq 1$ there exists $\varepsilon_k \in (0, k^{-2})$ such that

$$a_k^2 \int_0^\infty e^{-2bs} ds \int_{I\!R} [g_1(s, y-\varepsilon_k) - g_1(s,y)]^2 dy \leq 2^{-k}. \tag{5.5}$$

Let $x_k = k^{-1}$ and $z_k = k^{-1} + \varepsilon_k$. Then $z_k > x_k > z_{k+1} > x_{k+1} > \ldots$ decrease to zero. By (5.5) and the shift invariance of the Lebesgue measure it is easy to see that

$$\tilde{x}_n(s, \cdot) = \sum_{k=1}^n a_k [g_1(s, \cdot - z_k) - g_1(s, \cdot - x_k)], \quad s > 0,$$

defines a Cauchy sequence $\{\tilde{x}_n\} \subset \tilde{L}^2(I\!R)$ with limit $\tilde{x} \in \tilde{L}^2(I\!R)$ given by

$$\tilde{x}(s, \cdot) = \sum_{k=1}^\infty a_k [g_1(s, \cdot - z_k) - g_1(s, \cdot - x_k)], \quad s > 0. \tag{5.6}$$

To represent this element in the form of (5.3) we need to let

$$\mu = \sum_{k=1}^\infty a_k \delta_{z_k} - \sum_{k=1}^\infty a_k \delta_{x_k},$$

which is clearly not belonging to $S(I\!R)$ in general.

Now we consider the case where $(T_t)_{t\geq 0}$ is the transition semigroup of the absorbing barrier Brownian motion in $D = (0, \infty)$. Let $\gamma(dy) = (1 - e^{-y^2})dy$ and let $\tilde{L}^2(D, \gamma)$ be defined from $L^2(D, \gamma)$ and $(T_t)_{t\geq 0}$. Let

$$p_s(x,y) = g_1(s, y-x) - g_1(s, y+x), \quad s, x, y > 0, \tag{5.7}$$

and let

$$k_s(y) = 2^{-1}(d/dx) p_s(x,y)|_{x=0^+} = y g_1(s,y)/s, \quad s, y > 0. \tag{5.8}$$

Let $S(D, \gamma)$ be the set of signed-measures μ on D with total variation measures $|\mu|$ satisfying

$$\int_0^l ds \int_D \Big(\int_D p_s(x,y) |\mu|(dx) \Big)^2 \gamma(dy) < \infty \tag{5.9}$$

for some $l > 0$. Let $S_+(D, \gamma)$ and $\tilde{L}^2_+(D, \gamma)$ denote respectively the subsets of non-negative elements of $S(D, \gamma)$ and $\tilde{L}^2(D, \gamma)$.

Theorem 5.2 *There is a 1-1 correspondence between* $\tilde{x} \in \tilde{L}^2_+(D,\gamma)$ *and* $(a,\mu) \in [0,\infty) \times S_+(D,\gamma)$ *which is given by*

$$\tilde{x}(s,\cdot) = ak_s(\cdot) + \int_D p_s(z,\cdot)\mu(dz), \quad s > 0. \tag{5.10}$$

Proof. If $(a,\mu) \in [0,\infty) \times S_+(D,\gamma)$, then (5.10) defines an entrance path $\tilde{x}$ for $(T_t)_{t\geq 0}$. Since

$$\int_0^l k_s(y)^2 ds \leq \int_0^\infty \frac{y^2}{2\pi s^3} e^{-y^2/s} ds = \frac{1}{2\pi y^2},$$

we have

$$\begin{aligned} \int_0^l \|\tilde{x}(s,\cdot)\|^2 ds &= \int_0^l ds \int_D \Big(ak_s(y) + \int_D p_s(z,y)\mu(dz)\Big)^2 \gamma(dy) \\ &\leq 2a^2 \int_0^l ds \int_D k_s(y)^2 \gamma(dy) \\ &\quad +2 \int_0^l ds \int_D \Big(\int_D p_s(z,y)\mu(dz)\Big)^2 \gamma(dy) \\ &< \infty, \end{aligned}$$

that is, $\tilde{x} \in \tilde{L}^2_+(D,\gamma)$. Conversely, suppose that $\tilde{x} \in \tilde{L}^2_+(I\!R)$ and let $\kappa_s(dy) = \tilde{x}(s,y)dy$. Then $(\kappa_s)_{s>0}$ is a measure-valued entrance law for $(T_t)_{t\geq 0}$. By the property of the absorbing barrier Brownian motion, there is a constant $a \geq 0$ and a measure μ on D such that (5.10) holds; see e.g. [9, Lemma 1.1]. Since

$$\int_0^l ds \int_D \Big(\int_D p_s(z,y)\mu(dz)\Big)^2 \gamma(dy) \leq \int_0^l \|\tilde{x}(s,\cdot)\|^2 ds, < \infty.$$

we have $\mu \in S(D,\gamma)$. $\square$

By the general results of section 4, an OU-process associated with the absorbing barrier Brownian motion in $D = (0,\infty)$ always has càdlàg realization in $\bar{L}^2(D,\gamma) \supset \tilde{L}^2(D,\gamma)$ defined from $L^2(D,\gamma)$. It was observed in [**?**] that in a special case the process also has càdlàg realization in $S(D,\gamma)$.

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